The equation (2) is a recurrence relationship that leads to the factorial concept. First observe that if \( x = 1 \), then (1) can be evaluated, and in particular,

\[ \Gamma(1) = 1. \]

From (2)

\[ \Gamma(x + 1) = x\Gamma(x) = x(x - 1)\Gamma(x - 1) = \ldots x(x - 1)(x - 2) \ldots (x - k)\Gamma(x - k) \]

If \( x = n \), where \( n \) is a positive integer, then

\[ \Gamma(n + 1) = n(n - 1)(n - 2) \ldots 1 = n! \]  \hspace{1cm} (3)

If \( x \) is a real number, then \( x! = \Gamma(x - 1) \) is defined by \( \Gamma(x - 1) \). The value of this identification is in intuitive guidance.

If the recurrence relation (2) is characterized as a differential equation, then the definition of \( \Gamma(x) \) can be extended to negative real numbers by a process called analytic continuation. The key idea is that even though \( \Gamma(x) \) is defined in (1) is not convergent for \( x < 0 \), the relation \( \Gamma(x) = \frac{1}{x} \Gamma(x + 1) \) allows the meaning to be extended to the interval \(-1 < x < 0\), and from there to \(-2 < x < -1\), and so on.

The factorial notion guides us to information about \( \Gamma(x + 1) \) in more than one way. In the eighteenth century, Sterling introduced the formula (for positive integer values \( n \))
\[
\lim_{n \to \infty} \frac{\sqrt{2\pi n}^{n+1} e^{-n}}{n!} = 1\] (4)

This is called Sterling’s formula and it indicates that \( n! \) asymptotically approaches \( \sqrt{2\pi n}^{n+1} e^{-n} \) for large values of \( n \). This information has proved useful, since \( n! \) is difficult to calculate for large values of \( n \).

There is another consequence of Sterling’s formula. It suggests the possibility that for sufficiently large values of \( x \),

\[ x! = \Gamma(x + 1) \approx \sqrt{2\pi x}^{x+1} e^{-x} \] (5a)

It is known that \( \Gamma(x + 1) \) satisfies the inequality

\[ \sqrt{2\pi x}^{x+1} e^{-x} < \Gamma(x + 1) < \sqrt{2\pi x}^{x+1} e^{-x} e^{\frac{1}{12x(x+1)}} \] (5b)

Since the factor \( e^{\frac{1}{12x(x+1)}} \to 0 \) for large values of \( x \), the suggested value (5a) of \( \Gamma(x + 1) \) is consistent with (5b).

An exact representation of \( \Gamma(x + 1) \) is suggested by the following manipulation of \( n! \). (It depends on \( (n+k)! = (k+n)! \))

\[
n! = \lim_{k \to \infty} \frac{12 \ldots n(n+1)(n+2) \ldots (n+k)}{(n+1)(n+2) \ldots (n+k)} = \lim_{k \to \infty} \frac{k! k^n}{(n+1) \ldots (n+k)} \lim_{k \to \infty} \frac{(k+1)(k+2) \ldots (k+n)}{k^n}
\]

Since \( n \) is fixed the second limit is one, therefore, \( n! = \lim_{k \to \infty} \frac{k! k^n}{(n+1) \ldots (n+k)} \) (This must be read as an infinite product.)

This factorial representation for positive integers suggests the possibility that

\[
\Gamma(x + 1) = x! = \lim_{k \to \infty} \frac{k! k^x}{(x+1) \ldots (x+k)} \quad x \neq -1, -2, -k
\] (6)

Gauss verified this identification back in the nineteenth century.

This infinite product is symbolized by \( \prod(x, k) \), i.e., \( \prod(x, k) = \lim_{k \to \infty} \frac{k! k^x}{(x+1) \ldots (x+k)} \) It is called Gauss’s function and through this symbolism,

\[
\Gamma(x + 1) = \lim_{x \to \infty} \prod(x, k)
\] (7)

The expression for \( \frac{1}{\Gamma(x)} \) (which has some advantage in developing the derivative of \( \Gamma(x) \)) results as follows. Put (6a) in the form

\[
\lim_{k \to \infty} \frac{k^x}{(1 + x)(1 + \frac{x}{2}) \ldots (1 + \frac{x}{k})} \quad x \neq -\frac{1}{2}, -\frac{1}{3}, \ldots -\frac{1}{k}
\]
Next, introduce
\[ \gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \ln k \]

Then
\[ \gamma = \lim_{k \to \infty} \gamma_k \]
is Euler’s constant. This constant has been calculated to many places, a few of which are \( \gamma \approx 0.57721566 \ldots \)

By letting \( k^x = e^{x \ln k} = e^{x[-\gamma_k + 1 + \frac{1}{2} + \ldots + \frac{1}{k}]} \), the representation (6) can be further modified so that
\[
\begin{align*}
\Pi(x + 1) &= e^{-\gamma x} \lim_{k \to \infty} \frac{e^x}{1 + x} \frac{e^{x/2}}{1 + x/2} \ldots \frac{e^{x/k}}{1 + x/k} = e^{-\gamma x} \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + x/k} = \prod_{k=1}^{\infty} k^x k! (k + x) \\
&= \lim_{k \to \infty} \frac{1.2.3 \ldots k}{(x + 1)(x + 2) \ldots (x + k)} x^x = \lim_{k \to \infty} \prod_{k=1}^{\infty} \frac{1}{(x, k)} \quad (8)
\end{align*}
\]

Since \( \Pi(x + 1) = x \Pi(x) \)
\[
\frac{1}{\Pi(x)} = xe^{\gamma x} \lim_{k \to \infty} \frac{e^x}{1 + x} \frac{e^{x/2}}{1 + x/2} \ldots \frac{e^{x/k}}{1 + x/k} = xe^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k} \quad (9)
\]

Another result of special interest emanates from a comparison of \( \Pi(x) \Pi(x + 1) \) with the “famous” formula
\[
\frac{\pi x}{\sin \pi x} = \lim_{k \to \infty} \left\{ \frac{1}{1 - x^2} \frac{1}{1 - (x/2)^2} \ldots \frac{1}{1 - (x/k)^2} \right\} = \prod_{k=1}^{\infty} \left(1 - \frac{x}{k}\right)^2 \quad (10)
\]

\( \Pi(1 - x) \) is obtained from \( \Pi(y) = \frac{1}{y} \Pi(y + 1) \) by letting \( y = -x \), i.e.,
\[
\Pi(-x) = -\frac{1}{x} \Pi(1 - x) \quad \text{or} \quad \Pi(1 - x) = -x \Pi(-x)
\]

Now use (8) to produce
\[
\begin{align*}
\Pi(x) \Pi(1 - x) &= \left( x^{-1} e^{-\gamma x} \lim_{k \to \infty} \prod_{k=1}^{\infty} (1 + x/k)^{-1} e^{x/k} \right) \left( e^{\gamma x} \lim_{k \to \infty} \prod_{k=1}^{\infty} (1 - x/k)^{-1} e^{-x/k} \right) \\
&= \frac{1}{x} \lim_{k \to \infty} \prod_{k=1}^{\infty} (1 - (x/k)^2)
\end{align*}
\]

Thus
\[
\Pi(x) \Pi(1 - x) = \frac{\pi x}{\sin \pi x}, \quad 0 < x < 1 \quad (11a)
\]
Observe that (11) yields the result

\[ I\left(\frac{1}{2}\right) = \sqrt{\pi} \]  

(11b)

Another exact representation of \( I(x + 1) \) is

\[ I(x + 1) = \sqrt{2\pi} \Gamma(x + 1) e^{-x} \left\{ 1 + \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3} + \ldots \right\} \]  

(12)

The method of obtaining this result is closely related to Sterling’s asymptotic series for the gamma function.

The duplication formula is

\[ 2^{2x-1} I(x) \Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi} I(2x) \]  

(13a)

The duplication formula is a special case \( m = 2 \) of the following product formula:

\[ I(x) \Gamma\left(x + \frac{1}{m}\right) \Gamma\left(x + \frac{2}{m}\right) \ldots \Gamma\left(x + \frac{m-1}{m}\right) = m^{\frac{1}{2} - mx} (2\pi)^{\frac{m-1}{2}} I(mx) \]  

(13b)

It can be shown that the gamma function has continuous derivatives of all orders. They are obtained by differentiating (with respect to the parameter) under the integral sign.

It helps to recall that \( I(\infty) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt \) and that if \( y = t^{x-1} \), then \( \ln y = \ln t^{x-1} = (x - 1) \ln t \).

Therefore, \( \frac{1}{y} y' = \ln t \).

It follows that

\[ I''(x) = \int_{0}^{\infty} t^{x-1} e^{-t \ln t} \, dt. \]  

(14a)

This result can be obtained (after making assumptions about the interchange of differentiation with limits) by taking the logarithm of both sides of (9) and then differentiating.

In particular,

\[ I''(1) = -\gamma \quad (\gamma \text{ is the Euler's constant.}) \]  

(14b)

It also may be shown that

\[ \frac{I''(x)}{I(x)} = -\gamma + \left( \frac{1}{1 - \frac{1}{x}} \right) + \left( \frac{1}{2} - \frac{1}{x+1} \right) + \ldots \left( \frac{1}{x+n-1} \right) \]  

(15)

**Example 1:** Prove that \( I(1) = 1 \)

**Solution:**

\[ I(n) = \int_{0}^{\infty} x^{n-1} e^{-x} \, dx \]

Put \( n = 1 \),

\[ I(1) = \int_{0}^{\infty} e^{-x} \, dx = \left[ -e^{-x} \right]_{0}^{\infty} = 1 \quad \text{Proved} \]
Example 2: Prove that

(i) \( I(n + 1) = nI(n) \)

(ii) \( I(n + 1) = n! \)

Solution:

(i) \( I(n) = \int_0^{\infty} x^{n-1} e^{-x} \, dx \)

Integrating by parts, we have

\[
I(n) = \left[ x^{n-1} e^{-x} \right]_0^{\infty} - (n - 1) \int_0^{\infty} x^{n-2} e^{-x} \, dx
\]

\[
= \lim_{x \to 0} \frac{x^{n-1}}{e^x} = \lim_{x \to 0} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots + x^{n-1} = 0
\]

\[
= (n - 1) \int_0^{\infty} x^{n-2} e^{-x} \, dx
\]

\[
I(n) = (n - 1)I(n-1)
\]

--------(*)

\[
I(n + 1) = nI(n)
\]

Replacing \( n \) by \( (n + 1) \) \hspace{1cm} \text{Proved}

(ii) Replace \( n \) by \( (n - 1) \) in (*) we get

\[
I(n - 1) = (n - 2)I(n-2)
\]

Putting the value \( I(n - 1) \) in (*) we get

\[
I(n) = (n - 1)(n - 2)I(n-2)
\]

Similarly \( I(n) = (n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 \cdot I(1) \)

--------(**)

Putting the value of \( I(1) \) in (**) we have:

\[
I(n) = (n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 \cdot 1
\]

\[
I(n) = (n - 1)!
\]

Replacing \( n \) by \( (n + 1) \), we have

\[
I(n + 1) = n!
\]

\hspace{1cm} \text{Proved}

Example 3: Evaluate each of the following.

(a) \( \frac{I(6)}{2I(3)} = \frac{5!}{2 \cdot 2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2} = 30 \)

(b) \( \frac{\frac{3!}{2!}}{\frac{2!}{2!}} = \frac{3! \cdot 2!}{2! \cdot 2!} = \frac{3!}{2!} = \frac{3}{4} \)

(c) \( \frac{I(3)/I(2.5)}{I(5.5)} = \frac{21(1.5)(0.5)I(0.5)}{(4.5)(3.5)(2.5)(1.5)(0.5)I(0.5)} = \frac{16}{315} \)
Example 4: Evaluate \( \int_0^\infty \sqrt{x} e^{-\sqrt{x}} \, dx \)

**Solution:** Let \( I = \int_0^\infty x^{1/4} e^{-\sqrt{x}} \, dx \) ................ (1)

Putting \( \sqrt{x} = t \) or \( x = t^2, \, dx = 2tdt \) in (1), we get

\[
I = \int_0^\infty t^{1/2} e^{-t} \cdot 2 \, t \, dt = 2 \int_0^\infty t^{3/2} e^{-t} \, dt
\]

\[
= 2 \Gamma\left(\frac{5}{2}\right)
\]

By definition

\[
= 2 \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3}{2} \sqrt{\pi}
\]

Ans.

Example 5: Evaluate \( \int_0^\infty \sqrt{x} e^{-\sqrt{x}} \, dx \).

**Solution:** Let \( I = \int_0^\infty \sqrt{x} e^{-\sqrt{x}} \, dx \) ................ (1)

Putting \( \sqrt{x} = t \) or \( x = t^3, \, dx = 3t^2 \, dt \) in (1) we get

\[
I = \int_0^\infty t^{3/2} e^{-t} \cdot 3 \, t^2 \, dt = 3 \int_0^\infty t^{7/2} e^{-t} \, dt = 3 \Gamma\left(\frac{9}{2}\right)
\]

\[
= 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{315}{16} \sqrt{\pi}
\]

Ans.

Example 6: Evaluate \( \int_0^\infty x^{n-1} e^{-\frac{h^2}{2} x^2} \, dx \).

**Solution:** Let \( I = \int_0^\infty x^{n-1} e^{-\frac{h^2}{2} x^2} \, dx \) ................ (1)

Putting \( t = \frac{h^2 x^2}{h} \), or \( x = \frac{\sqrt{t}}{h}, \, dx = \frac{dt}{2h\sqrt{t}} \)

Thus (1) becomes

\[
I = \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t} \, \frac{dt}{2h\sqrt{t}}
\]

\[
= \frac{1}{2h^2} \int_0^\infty t^{n-1} e^{-t} \, \frac{dt}{\sqrt{t}} = \frac{1}{2h^2} \int_0^\infty t^{n-2} e^{-t} \, dt
\]

\[
= \frac{1}{2h^2} \Gamma\left(\frac{n}{2}\right)
\]

Ans.

Example 7: Prove that \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \).

**Solution:** \( \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt \).

Letting \( t = x^2 \) thus integral becomes \( 2 \int_0^\infty e^{-x^2} \, dx \)
Let $I_M = \int_0^M e^{-x^2} \, dx = \int_0^M e^{-y^2} \, dy$ let $\lim_{M \to \infty} I_M = I$, the required value of the integral. Then

\[
I_M^2 = (\int_0^M e^{-x^2} \, dx)(\int_0^M e^{-y^2} \, dy)
\]

\[
= \int_0^M \int_0^M e^{-(x^2+y^2)} \, dxdy
\]

\[
= \iint_R e^{-(x^2+y^2)} \, dxdy
\]

Where $R_M$ is the square $OACE$ of side $M$ (see Fig.2). Since integrand is positive, we have

\[
\iint_R e^{-(x^2+y^2)} \, dxdy \leq I_M \leq \iint_R e^{-(x^2+y^2)} \, dxdy
\]

Where $R_1$ and $R_2$ are the regions in the first quadrant bounded by the circles having radii $M$ and $M\sqrt{2}$, respectively.

Using polar coordinates, we have from (1),

\[
\int_0^{\pi/2} \int_0^M e^{-\rho^2} \rho \, d\rho d\phi \leq I_M \leq \int_0^{\pi/2} \int_0^{M\sqrt{2}} e^{-\rho^2} \rho \, d\rho d\phi
\]

Or

\[
\frac{\pi}{4} (1 - e^{-M^2}) \leq I_M \leq \frac{\pi}{4} (1 - e^{-2M^2})
\]

Then taking the limit as $M \to \infty$ in (3), we find

\[
\lim_{M \to \infty} I_M^2 = I^2 = \frac{\pi}{4} \quad \text{and} \quad I = \sqrt{\pi}/2
\]

Then

\[
\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-x^2} \, dx = 2 \left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi} \quad \text{Proved}
\]

1. Evaluate each integral:
   a) $\int_0^\infty x^4 e^{-x} \, dx$
   b) $\int_0^\infty x^6 e^{-3x} \, dx$
   c) $\int_0^\infty x^2 e^{-2x^2} \, dx$
   d) $\int_0^\infty \sqrt{y} e^{-y^2} \, dy$
   e) $\int_0^\infty 3 - 4x^2 \, dx$
   f) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$
2. Evaluate $\int_0^\infty x^m e^{-ax^n} \, dx$ where $m, n, a$ are positive constants.

3. Evaluate:
   a) $\Gamma(-1/2)$
   b) $\Gamma(-5/2)$

4. Evaluate $\int_0^\infty \frac{x^a}{a^x} \, dx$

5. Evaluate $\int_1^1 x^{n-1} \left[ \ln \left( \frac{1}{x} \right) \right]^{m-1} \, dx$

**THE BETA FUNCTION**

The beta function is a two-parameter composition of gamma functions that has been useful enough in application to gain its own name. Its definition is:

$$B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt \quad (1)$$

If $x \geq 1$ and $y \geq 1$, this is a proper integral. If $x > 0, y > 0$ and either or both $x < 1$ or $y < 1$, the integral is improper but convergent.

It is shown in Example No.2 that the beta function can be expressed through gamma functions in the following way

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad (2)$$

Many integrals can be expressed through beta and gamma functions. Two of special interest are:

$$\int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta \, d\theta = \frac{1}{2} B(x, y) = \frac{1}{2} \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad (3)$$

$$\int_0^\infty \frac{x^{p-1}}{1+x} \, dx = \Gamma(p) \Gamma(p-1) = \frac{\pi}{\sin \pi p} \quad 0 < p < 1 \quad (4)$$

**Example 1:** Prove that
(a) $B(u, v) = B(v, u)$  
(b) $B(u, v) = 2 \int_0^\pi \sin^{2u-1} \theta \cos^{2v-1} \theta \, d\theta$

**Solution:**

a) Using the transformation $x = 1 - y$, we have

$$B(u, v) = \int_0^1 x^{u-1} (1 - x)^{v-1} \, dx = \int_0^1 (1 - y)^{u-1} (y)^{v-1} \, dy = \int_0^1 (y)^{v-1} (1 - y)^{u-1} \, dy = B(v, u)$$

b) Using the transformation $x = \sin^2 \theta$, we have

$$B(u, v) = \int_0^1 x^{u-1} (1 - x)^{v-1} \, dx = \int_0^\pi \sin^2 \theta \cos^{u-1} \theta (\cos^2 \theta)^{v-1} \sin \theta \cos \theta \, d\theta$$

$$= 2 \int_0^\pi \sin^{2u-1} \theta \cos^{2v-1} \, d\theta$$

**Example 2:** Prove that $B(u, v) = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u+v)}$, $u, v > 0$
Solution:
Letting \( z = x^2 \), we have \( I(u) = \int_0^\infty z^{u-1} e^{-z} \, dz = 2 \int_0^\infty x^{2u-1} e^{-x^2} \, dx \).

Similarity, \( I(v) = 2 \int_0^\infty y^{2v-1} e^{-y^2} \, dy \). Then

\[
I(u)I(v) = 4 \left( \int_0^\infty x^{2u-1} e^{-x^2} \, dx \right) \left( \int_0^\infty y^{2v-1} e^{-y^2} \, dy \right) = 4 \int_0^\infty \int_0^\infty x^{2u-1} y^{2v-1} e^{-(x^2+y^2)} \, dx \, dy
\]

Transforming to polar coordinates, \( x = \rho \cos \theta \), \( y = \rho \sin \theta \)

\[
I(u)I(v) = 4 \int_0^{\pi/2} \int_0^\infty \rho^{2(u+v)-1} e^{-\rho^2} \, d\rho \, d\theta \int_0^{\pi/2} \cos^{2u-1} \theta \sin^{2v-1} \theta \, d\theta
\]

\[
= 4 \left( \int_0^{\pi/2} \cos^{2u-1} \theta \, d\theta \right) \left( \int_0^{\pi/2} \sin^{2v-1} \theta \, d\theta \right)
\]

\[
= 2I(u+v) \left( \int_0^{\pi/2} \cos^{2u-1} \theta \, d\theta \right) = I(u+v)B(u,v)
\]

Thus,

\[
B(u,v) = \frac{I(u)I(v)}{I(u+v)}
\]

Proved

From example No.1 and No.2 we show that:

\[
\int_0^{\pi/2} \sin^{2u-1} \theta \cos^{2v-1} d\theta = \frac{I(u)I(v)}{2I(u+v)}
\]

Example 3: Evaluate each of the following integrals.

a) \( \int_0^1 x^4(1-x)^3 \, dx = B(5,4) = \frac{I(5)I(4)}{I(9)} = \frac{4!3!}{8!} = \frac{1}{280} \)

b) \( \int_0^2 \frac{x^2 \, dx}{\sqrt{2-x}} \). Letting \( x = 2v \), the integral becomes

\[
4\sqrt{2} \int_0^1 v^2 \left( 1 - v^2 \right)^{1/2} \, dv = 4\sqrt{2} B \left( 3,1/2 \right) = \frac{4\sqrt{2} \pi \Gamma(3) \Gamma(1/2)}{\Gamma(3/2)} = \frac{64\sqrt{2}}{15}
\]

c) \( \int_0^a y^4 \sqrt{a^2 - y^2} \, dy \). Letting \( y^2 = a^2 x \) or \( y = a\sqrt{x} \), the integral becomes

\[
a^6 \int_0^1 x^{3/2}(1-x)^{1/2} \, dx = a^6 B(5/2,3/2) = \frac{a^6 \pi \Gamma(5/2) \Gamma(3/2)}{\Gamma(4)} = \frac{\pi a^6}{16}
\]

Example 4: Evaluate (a) \( \int_0^{\pi/2} \sin^6 \theta \, d\theta \) \quad (b) \( \int_0^{\pi/2} \sin^4 \theta \cos^5 \theta \, d\theta \) \quad (c) \( \int_0^\pi \cos^4 \theta \, d\theta \)

a) Let \( 2u - 1 = 6,2v - 1 = 0 \) i.e. \( u = 7/2, v = 1/2 \)

Then the required integral has the value \( \frac{I(7/2)I(1/2)}{2I(4)} = \frac{5\pi}{32} \)

b) Letting \( 2u - 1 = 4, 2v - 1 = 5 \), the required integral has the value \( \frac{I(5/2)I(3)}{2I(11/2)} = \frac{8}{315} \)
c) The given integral \( = 2 \int_0^{\pi/2} \cos^4 \theta \, d\theta \)

Thus letting \( 2u - 1 = 0 \), \( 2v - 1 = 4 \), the required integral has the value \( \frac{2I(1/2)I(5/2)}{2I(3)} = \frac{3\pi}{8} \)

**Homework**

1. Evaluate (a) \( B(3,5) \) (b) \( B(3/2, 2) \) (c) \( B(1/3, 2/3) \)

2. Find:
   a) \( \int_0^1 x^2 (1 - x)^3 \, dx \)
   b) \( \int_0^1 x^4 (1 - \sqrt{x})^5 \, dx \)
   c) \( \int_0^1 (1 - x^3)^{-1/2} \, dx \)
   d) \( \int_0^1 \sqrt{(1-x)/x} \, dx \)
   e) \( \int_0^2 (4 - x^2)^{3/2} \, dx \)

3. Evaluate:
   a) \( \int_0^4 u^{3/2} (4 - u)^{5/2} \, du \)
   b) \( \int_0^3 \frac{dx}{\sqrt{3x-x^2}} \)
   c) \( \int_0^{\pi/2} \sin^4 \theta \cos^4 \theta \, d\theta \)
   d) \( \int_0^{2\pi} \cos^6 \theta \, d\theta \)